# Experiments on the growth of groups (Intelligence of Low-dimensional Topology)

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# Experiments on the growth of groups

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## 1 Introduction

I gave a talk at the conference "Intelligence of Low-dimensional Topology, 2012". I discussed some known results on the growth of groups, from the view point of geometric group theory. Following computer experiments on examples, we try to raise questions on knot groups. This is a brief report from that talk. The computer experiments are done by Yasushi Yamashita, using KBMAG [10]. I am not a specialist on the subject, and benefited much from talking to M.Davis, R.Kellerhals and T.Nagnibeda.

### 1.1 Growth function

Let $G$ be a group with a finite generating set $S$. For $g\in G$, let $|g|$ be the word length with respect to $S$. Define

$$a_n=\sharp\{g\in G||g|=n\}.$$

The *growth function* is defined by

$$\gamma_{G,S}(t)=\sum_{n}a_nt^n.$$

It is easy to compute for free groups and free abelian groups with standard generators, but in general, it is very difficult to compute $a_n$ and $\gamma_{G,S}(t)$.

Here are more complicated examples. Serre found that $\gamma_{G,S}(t)$ is a rational function when $(G,S)$ is a Coxeter group with the standard generators (cf. [5]). As an easy example, for $G=\langle a,b|a^2,b^2\rangle$, infinite dihedral group,

$$\gamma(t)=\frac{(1+t)^2}{1-t^2}.$$

$\gamma_{G,S}(t)$ is rational for $N=\langle x,y|[[x,y],x],[[x,y],y]\rangle$, Heisenberg group, which is a nilpotent group.

$$\gamma(t)=\frac{t^8+9t^7+6t^6+21t^5+8t^4+11t^3+4t^2+t+1}{(t-1)^4(t^2+1)(t^2+t+1)},$$

and $a_n \sim n^3$ (Shapiro [12]).

$\gamma_{G,S}(t)$ is rational for surface groups with the standard generators (Cannon [2]). For example, for $G = \langle a, b, c, d | [a, b][c, d] \rangle$,

$$\gamma(t) = \frac{1 + 2t + 2t^2 + 2t^3 + t^4}{1 - 6t - 6t^2 - 6t^3 + t^4}.$$

Notice that $\gamma(t) = \gamma(1/t)$, which is called *reciprocity*, [8].

## 2 Automatic groups and hyperbolic groups

There is a theorem which explains the rationality for a large class of groups. See the book [7] for precise definitions and statements.

**Theorem 2.1 (Epstein and others [7])** *If $(G, S)$ is an automatic group (with a regular language) by geodesics, then $\gamma_{G,S}(t)$ is rational. The automaton computes $a_n$.*

For example, the theorem applies to surface groups with the standard presentations. A Coxeter group $(G, S)$ is an automatic group by geodesics (Brick-Howlett [1], Davis-Shapiro [6]). It explains the rationality of $\gamma(t)$. An Artin group $(G, S)$ of finite type (for example, Braid groups) has an automatic structure (w.r.t. the generating set of simple divisors), therefore $\gamma_{G,S}(t)$ is rational (Charney-Meier [3]).

Another example is $G = \langle a, b | aba = bab \rangle$, the Trefoil knot group and the Braid group $B_3$,

$$\gamma(t) = \frac{1 - 2t - 7t^2 + 2t^3 + 12t^4}{(1 - t)(1 - 2t)(1 - 3t)(1 - 4t)}.$$

Here is a large class of examples which the theorem applies to.

**Theorem 2.2 (cf [7])** *A (word-)hyperbolic group $G$ is automatic by geodesics for any generating set $S$, therefore $\gamma_{G,S}(t)$ is rational.*

The rationality of the growth function depends on a set of generators in general. The significance of the theorem is that it is true for all generators.

For example, the fundamental group of a closed hyperbolic manifold/orbifold is word-hyperbolic. If $G$ contains $\mathbb{Z}^2$, then it is not hyperbolic. In particular, (hyperbolic) knot groups are not hyperbolic.

**Question 2.3** *(Hyperbolic) knot groups are not hyperbolic groups, but is $\gamma_{G,S}(t)$ rational for some/any $S$ ?*

### 2.1 Experiments

There is a program [10] which seeks for an automatic structure by geodesics if a presentation of a group is given. We used this program in the following computation of growth functions. In the following exampls, the growth functions are rational, but not reciprocal.

- trefoil knot: $\langle a, b | aa = bbb \rangle$, as (2,3)-torus knot.
  $\frac{(x+1)\left(4x^7+6x^6+6x^5-10x^4-7x^3+x^2+2x+1\right)}{(x-1)(2x^2-1)^2(x^3+x^2-1)} = 1+4x+12x^2+22x^3+40x^4+66x^5+106x^6+168x^7+258x^8+\cdots$

- trefoil knot: $\langle a, b | aba = bab \rangle$, as an braid group $B_3$.
  $\frac{(x+1)\left(2x^3-x^2+x-1\right)}{(x-1)(2x-1)(x^2+x-1)} = 1+4x+12x^2+30x^3+68x^4+148x^5+314x^6+656x^7+1356x^8+\cdots$

- trefoil knot: $\langle a, b, c | cac^{-1}b^{-1}, aba^{-1}c^{-1} \rangle$, Wirtinger presentation.
  $\frac{(x+1)\left(2x^2-1\right)}{(x-1)(2x-1)^2} = 1+6x+20x^2+54x^3+134x^4+318x^5+734x^6+1662x^7+3710x^8+\cdots$

- figure eight knot: $\langle a, b | a^{-1}bab^{-1}aba^{-1}b^{-1}ab^{-1} \rangle$
  $\frac{(x+1)\left(2x^{10}-4x^9+x^8-4x^7+9x^6-9x^5+4x^4-3x^3+4x^2-3x+1\right)}{(x-1)(2x^{10}-5x^8+2x^7-3x^6+5x^5-6x^4+7x^3-8x^2+5x-1)} = 1+4x+12x^2+36x^3+108x^4+314x^5+900x^6+2580x^7+7396x^8+\cdots$

- figure eight knot: $\langle a, b, c, d | dbd^{-1}a^{-1}, aba^{-1}c^{-1}, bdb^{-1}c^{-1} \rangle$, Wirtinger presentation.
  $\frac{(x+1)\left(x^2-x-1\right)}{(x-1)(3x^2-5x+1)} = 1+8x+40x^2+178x^3+772x^4+3328x^5+14326x^6+61648x^7+265264x^8+\cdots$

## 3 Hyperbolic Coxeter groups

We recommned the book [5] as a reference of this section. Roughly speaking, $n$-dimensional *Hyperbolic* Coxeter groups are the ones which are *realized* by reflections along hyperplanes in $\mathbb{H}^n$. Each of them gives a hyperbolic orbifold. Those ones which are compact or of finite volume (and non-compact) are particularly interesting, but producing examples and the classification are hard, except for $n = 2, 3$.

The hyperbolic ones of compact quotients are word-hyperbolic, therefore, the growth functions are rational for any generating set. The finite volume ones are not hyperbolic, therefore we do not know if the growth functions are rational in general, although we do know for the Coxeter generators.

### 3.1 2 and 3 dimensional hyperbolic Coxeter group

Here are 2-dimensional and 3-dimentional examples:

- (2,3,7), 2-dim, compact,

$$\langle a, b, c | a^2, b^2, c^2, (ab)^2, (bc)^3, (ca)^7 \rangle,$$

  known to have smallest volume ($\pi/42$) among 2-dim, compact hyperbolic orbifolds (Siegel).

- $(2, 3, \infty)$: 2-dim, non-compact, finite volume.

$$\langle a, b, c | a^2, b^2, c^2, (ab)^2, (bc)^3 \rangle,$$

  known to have smallest volume ($\pi/6$) among 2-dim, hyperbolic, non-compact, orbifolds.

- $(3, 5, 3, 2)$, 3-dim, compact, its $\mathbb{Z}_2$-extension is expected to have smallest volume among all compact hyperbolic 3-orbifolds,

$$\langle a, b, c, d | a^2, b^2, c^2, d^2, (ab)^3, (bc)^5, (cd)^3, (ad)^2 \rangle.$$

- $(3, 3, 6, 2)$: 3-dim, non-compact, finite volume, smallest among non-compact, hyperbolic 3-orbifolds (Meyerhoff).

$$\langle a, b, c, d | a^2, b^2, c^2, d^2, (ab)^3, (bc)^3, (cd)^6, (ad)^2 \rangle.$$

### 3.2 Experiments on growth functions

- $\langle a, b, c | a^2, b^2, c^2, (ab)^2, (bc)^3, (ca)^7 \rangle$, 2-dim, compact.
  $\frac{(x+1)^2 \left(x^2+x+1\right) \left(x^6+x^5+x^4+x^3+x^2+x+1\right)}{x^{10}+x^9-x^7-x^6-x^5-x^4-x^3+x+1} = 1 + 3\,x + 5\,x^2 + 7\,x^3 + 9\,x^4 + 12\,x^5 + 16\,x^6 + 20\,x^7 + 24\,x^8 + \cdots$

  This is reciprocal. Of course, the reciprocity is sensitive to the generators. For example, if we add a generator $d = acacb$, then the reciprocity does not hold. Interestingly, if we add $d = abc$, the reciprocity holds.

  **Question 3.1** *For which generators does the reciprocity hold ? (cf. [8])*

- $\langle a, b, c | a^2, b^2, c^2, (ab)^2, (bc)^3 \rangle$, 2-dim, finite volume, not reciprocal.
  $\frac{x^4+3\,x^3+4\,x^2+3\,x+1}{-x^3-x^2+1} = 1 + 3\,x + 5\,x^2 + 7\,x^3 + 9\,x^4 + 12\,x^5 + 16\,x^6 + 21\,x^7 + 28\,x^8 + \cdots$

- $\langle a,b,c,d|a^2,b^2,c^2,d^2,(ab)^3,(bc)^5,(cd)^3,(ad)^2\rangle$, 3-dim, compact, reciprocal.
  $\frac{(x+1)^2\left(x^2+x+1\right)\left(x^4+x^3+x^2+x+1\right)}{x^8-3\,x^6-5\,x^5-5\,x^4-5\,x^3-3\,x^2+1} = 1+4\,x+11\,x^2+28\,x^3+70\,x^4+175\,x^5+436\,x^6+1086\,x^7+2706\,x^8+\cdots$

- $\langle a,b,c,d|a^2,b^2,c^2,d^2,(ab)^3,(bc)^3,(cd)^6,(ad)^2\rangle$, 3-dim, non-compact, finite volume, reciprocal.
  $\frac{(x+1)^2\left(x^2-x+1\right)\left(x^2+x+1\right)}{x^6-2\,x^5-x^4-x^2-2\,x+1} = 1+4\,x+11\,x^2+28\,x^3+70\,x^4+176\,x^5+441\,x^6+1104\,x^7+2764\,x^8+\cdots$

### 3.3 Growth rate

Define the (exponential) *growth rate* of $(G,S)$ by

$$r_{G,S} = \liminf_{n\to\infty} a_n^{1/n}.$$

For example, $r=0$ if $G$ is abelian, and $r=3$ if $(G,S)$ is a free group freely generated by $a,b$ and $S=\{a,b\}$.

If $\gamma_{G,S}(t)$ is rational, let $\{p_i\}$ be the set of poles. Then $1/r = \min_i |p_i|$.

The Coxeter group $\langle a,b,c|a^2,b^2,c^2,(ab)^2,(bc)^3,(ca)^7\rangle$ has smallest volume among all 2-dimensional hyperbolic, compact orbifolds. It also has the smallest growth rate among all 2-dim hyperbolic, compact, Coxeter groups (with respect to $S$), (E. Hironaka [9]). The Coxeter group $\langle a,b,c|a^2,b^2,c^2,(ab)^2,(bc)^3\rangle$ has smallest volume/growth rate among non-compact and finite volume ones in the same sense (Floyd).

There are no results which directly relates the smallest growth rate and volume. It only suggests a candidate to each other. In general, the growth rate depends on the generators.

**Question 3.2** *Does $S$ give the smallest growth rate in each case ?*

Figure 8 knot group $K$ has smallest hyperbolic volume among all hyperbolic knots (in fact all orientable cusped hyperbolic 3-manifolds), (Cao-Meyerhoff).

**Question 3.3** *Does $K$ has smallest growth rate among knots (with respect to "standard" generating set in certain sense, or after taking* inf *among all generating sets) ?*

For closed hyperbolic 3-manifolds, the volume map $M \mapsto \mathrm{volume}(M)$ is finite to one and its image is a well-ordered set.

**Question 3.4** *How about for the maps $M \mapsto r_{\pi_1(M)}$ and $M \mapsto \gamma_{\pi_1(M)}$ ?*

Again, we need to specify a generating set $S$.

## 4 Accumulation points

**Theorem 4.1 (Coornaert [4])** *Let $(G, S)$ be a hyperbolic group. Then there exist $A, B, C$ such that for any $n$*

$$Ae^{Cn} \leq a_n \leq Be^{Cn} \tag{1}$$

Notice $r_{G,S} = e^C$. By the theorem, for each $n$, $a_n/e^{Cn} \in [A, B]$. Therefore, there must be accumulation points in $[A, B]$.

**Question 4.2 (K.Saito [11])** *Under (1), are there only finitely many accumulation points ? Is it only one ?*

Saito noticed that $PSL(2, \mathbb{Z}) = \mathbb{Z}_2 * \mathbb{Z}_3$ has two accumulation points.

**Question 4.3** *Does (1) hold for (hyperbolic) knot groups ? If so, are there only finitely many/only one accumulation points ?*

The question concerns $a_n/e^{Cn}$. We did an experiment on the sequence $a_{n+1}/a_n$ for the figure 8 knot group: $G = \langle a, b | a^{-1}bab^{-1}aba^{-1}b^{-1}ab^{-1} \rangle$.
$\gamma(x) = \frac{2x^{11}-2x^{10}-3x^9-3x^8+5x^7-5x^5+x^4+x^3+x^2-2x+1}{2x^{11}-2x^{10}-5x^9+7x^8-5x^7+8x^6-11x^5+13x^4-15x^3+13x^2-6x+1} = 1+4x+12x^2+36x^3+108x^4+314x^5+900x^6+2580x^7+7396x^8+\ldots$

The smallest pole is $p = 0.349145768431\cdots$, therefore, $r = 1/p = 2.864133237225887\cdots$.

Computation of $a_{n+1}/a_n, n = 0, 1, \cdots$ seems to converge to $r$, which suggests that there is only one accumulation point:

$4.0, 3.0, 3.0, 3.0,$
$2.90740740741\ldots, 2.86624203822\ldots, 2.86666666667\ldots, 2.86666666667\ldots,$
$2.86479177934\ldots, 2.86416839721\ldots, 2.86418613848\ldots, 2.86417821144\ldots,$
$2.86414108951\ldots, 2.86413200513\ldots, 2.86413396932\ldots, \cdots$

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